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LETTER TO THE EDITOR

The quantum Gelfand–Levitan–Marchenko equations and form factors in the sine–Gordon model

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Abstract. The quantum Gelfand–Levitan–Marchenko equation obtained by the author in a previous work is used to calculate the form factors in the quantum sine–Gordon model.

The calculation of the Green functions for the models with a complicated vacuum structure is one of the most challenging problems in the theory of quantum completely integrable models (Faddeev 1980). In a number of papers (Lenard 1966, Thacker 1982, Jimbo *et al* 1980) the Green functions have been calculated for some special values of coupling constants when the models under consideration are equivalent to the free fermion ones. The behaviour of the Green functions in the neighbourhood of these points has been also investigated (Thacker 1982).

The greatest progress was achieved by Korepin (1984). The expansion was obtained of the two-particle Green function into an absolutely convergent series for the one-dimensional Bose gas with non-zero density.

In a paper by Karowsky and Weisz (1978) attempts were made to calculate the Green functions using some properties of form factors (matrix elements of local fields). The knowledge of all form factors makes it possible to calculate the Green functions.

In the paper mentioned above some relationships for form factors were obtained. These relationships follow from CPT invariance and crossing symmetry and, being purely kinematical, do not allow the calculation of form factors completely.

In the present paper the method of calculation of form factors in the sine–Gordon model with Hamiltonian

$$H=\int \{\tfrac{1}{2}u_t^2+\tfrac{1}{2}u_x^2+(m^2/8\gamma)[1-\cos 2(2\gamma)^{1/2}u]\}\,\mathrm{d}x$$

will be presented. Our method is based on the Gelfand–Levitan–Marchenko equations obtained by Smirnov (1984). The form factors of $\exp[\pm\mathrm{i}(2\gamma)^{1/2}u(x)]$ corresponding to the states with an arbitrary number of elementary bosons and their bound states will be obtained.

The investigation of form factors corresponding to solitons will be carried out in another paper.

The sine–Gordon model was quantised by Faddeev *et al* (1980). In the derivation of the Gelfand–Levitan–Marchenko equations (Smirnov 1984) we used the lattice version of the model (Izergin and Korepin 1981). In the previous paper (Smirnov 1984) the operators of creation–annihilation of physical excitations $\mathscr{R}_m^+(\alpha)$, $\mathscr{R}_m(\alpha)$

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(m-particle bound state of elementary bosons) and $\mathscr{H}^+(\alpha)$, $\mathscr{H}(\alpha)$ (holes) were used (+ does not mean adjoint!) These operators satisfy the Zamolodchikov–Faddeev algebra:

$$\mathscr{R}_m^+(\alpha)\mathscr{R}_n^+(\beta) = S_{mn}(\alpha-\beta)\mathscr{R}_n^+(\beta)\mathscr{R}_m^+(\alpha)$$

$$\mathscr{R}_m^+(\alpha)\mathscr{H}^+(\beta) = S_{mh}(\alpha-\beta)\mathscr{H}^+(\beta)\mathscr{R}_m^+(\alpha)$$

$$\mathscr{H}^+(\alpha)\mathscr{H}^+(\beta) = S_{hh}(\alpha-\beta)\mathscr{H}^+(\beta)\mathscr{H}^+(\alpha)$$

$$\mathscr{R}_m(\alpha)\mathscr{R}_n(\beta) = S_{mn}(\alpha-\beta)\mathscr{R}_n(\beta)\mathscr{R}_m(\alpha)$$

$$\mathscr{R}_m(\alpha)\mathscr{H}(\beta) = S_{mh}(\alpha-\beta)\mathscr{H}(\beta)\mathscr{R}_m(\alpha)$$

$$\mathscr{H}(\alpha)\mathscr{H}(\beta) = S_{hh}(\alpha-\beta)\mathscr{H}(\beta)\mathscr{H}(\alpha)$$

$$\mathscr{R}_m(\alpha)\mathscr{R}_n^+(\beta) = S_{nm}(\beta-\alpha)\mathscr{R}_n^+(\beta)\mathscr{R}_m(\alpha) + 2\pi\tan(\gamma pm/2)p^{-1}\delta_{mn}\delta(\alpha-\beta)$$

$$\mathscr{R}_m(\alpha)\mathscr{H}^+(\beta) = S_{hm}(\beta-\alpha)\mathscr{H}^+(\beta)\mathscr{R}_m(\alpha)$$

$$\mathscr{H}(\alpha)\mathscr{H}^+(\beta) = S_{hh}(\beta-\alpha)\mathscr{H}^+(\beta)\mathscr{H}(\alpha) + 2\pi p^{-1}\delta(\alpha-\beta)$$

where $p = \pi/(\pi-\gamma)$

$$S_{mn}(\sigma) = \prod_{j=1}^{m}\prod_{k=1}^{n} \frac{\sinh p[\sigma + \frac{1}{2}\mathrm{i}\gamma(m-n-2j+2k)] + \mathrm{i}\sin\gamma p}{\sinh p[\sigma + \frac{1}{2}\mathrm{i}\gamma(m-n-2j+2k)] - \mathrm{i}\sin\gamma p}$$

$$S_{mh}(\sigma) = \prod_{j=1}^{m} \frac{\mathrm{i}\cos\frac{1}{2}\gamma p + \sinh p[\sigma - \frac{1}{2}\mathrm{i}\gamma(m+1-2j)]}{\mathrm{i}\cos\frac{1}{2}\gamma p - \sinh p[\sigma - \frac{1}{2}\mathrm{i}\gamma(m+1-2j)]}$$

$$S_{hh}(\sigma) = -\exp\left(-\mathrm{i}\int_0^\infty \frac{\sinh(\frac{1}{2}\pi-\gamma)k}{k\cosh\frac{1}{2}(\pi-\gamma)k\sinh(\frac{1}{2}\gamma)k}\sin k\sigma\,\mathrm{d}k\right).$$

With the accurate thermodynamical limit (Destry and Lowenstein 1982) in the formulae for norms of Bethe's vectors (Korepin 1982), it can be proved that the normalised 'in' state of the bound states of $n_1, n_2, \ldots, n_k$ elementary bosons with momenta $p_1, \ldots, p_k$ ($p_i \equiv (w_i, k_i) = 2m_r\sin(\frac{1}{2}\gamma n_i p)(\cosh p\sigma_i, \sinh p\sigma_i)$; m_r is the mass of the soliton) has the following representation in terms of the operators $\mathscr{R}_m^+$:

$$|p_1, n_1; \ldots; p_k, n_k\rangle_{\mathrm{in}} = N_{n_1}^{-1}\ldots N_{n_k}^{-1}\mathscr{R}_{n_{j_1}}^+(\sigma_{j_1})\ldots\mathscr{R}_{n_{j_k}}^+(\sigma_{j_k})\Omega_{\mathrm{ph}} \qquad (1)$$

where $\sigma_{j_1} > \ldots > \sigma_{j_k}$, Ω_{ph} is the physical vacuum,

$$N_m = (2\pi\sin\gamma pm)^{1/2}\exp\left(2\int_0^\infty \frac{\cosh(\frac{1}{2}\gamma k)\sinh^2(\frac{1}{2}\gamma mk)}{k\sinh(\frac{1}{2}\gamma k)\cosh^2\frac{1}{2}(\pi-\gamma)k}\,\mathrm{d}k\right)$$

$${}_{\mathrm{in}}\langle p_1', n_1'; \ldots; p_k', n_k'|p_1, n_1; \ldots; p_k, n_k\rangle_{\mathrm{in}}$$
$$= \delta_{n_1,n_1'}\ldots\delta_{n_k,n_k'}\delta(p\sigma_1 - p\sigma_1')\ldots\delta(p\sigma_k - p\sigma_k').$$

The similar formulae for the 'in' state can be found in the papers by Faddeev (1980) and Faddeev and Takhtajan (1982).

Consider the operators $\psi_m(\sigma)(1 \leqslant m \leqslant [\pi/\gamma]-1)$, $\psi_h(\sigma)$ which leave the vacuum invariant and have the following commutative relations with $\mathscr{R}_n^+$, $\mathscr{H}^+$:

$$\psi_m(\sigma_1)\mathscr{R}_n^+(\sigma_2) = \xi_{mn}(\sigma_1-\sigma_2)\mathscr{R}_n^+(\sigma_2)\psi_m(\sigma_1)$$

$$\psi_h(\sigma_1)\mathscr{R}_n^+(\sigma_2) = \xi_{hm}(\sigma_1-\sigma_2)\mathscr{R}_n^+(\sigma_2)\psi_h(\sigma_1)$$

$$\psi_m(\sigma_1)\mathcal{H}^+(\sigma_2)=\xi_{mh}(\sigma_1-\sigma_2)\mathcal{H}^+(\sigma_2)\psi_m(\sigma_1)$$

$$\psi_h(\sigma_1)\mathcal{H}^+(\sigma_2)=\xi_{hh}(\sigma_1-\sigma_2)\mathcal{H}^+(\sigma_2)\psi_h(\sigma_1)$$

where

$$\xi_{mn}(\sigma)=\exp\left(-2\int_0^\infty \frac{\cos k\sigma \sinh\frac{1}{2}\pi k \sinh\frac{1}{2}\gamma m k \sinh\frac{1}{2}\gamma n k}{k\sinh(\pi-\gamma)k\sinh\frac{1}{2}\gamma k\cosh\frac{1}{2}(\pi-\gamma)k}\,\mathrm{d}k\right)$$

$$\xi_{mh}(\sigma)=\exp\left(-\int_{-\infty}^\infty \frac{\exp[-\mathrm{i}k\sigma+\frac{1}{2}(\pi-\gamma)k]\sinh\frac{1}{2}\pi k\sinh\frac{1}{2}\gamma m k}{2k\sinh(\pi-\gamma)k\cosh\frac{1}{2}(\pi-\gamma)k\sinh\frac{1}{2}\gamma k}\,\mathrm{d}k\right)$$

$$\xi_{hm}(\sigma)=\exp\left(\int_{-\infty}^\infty \frac{\exp[-\mathrm{i}k\sigma-\frac{1}{2}(\pi-\gamma)k]\sinh\frac{1}{2}\pi k\sinh\frac{1}{2}\gamma m k}{2k\sinh(\frac{1}{2}\gamma)k\sinh(\pi-\gamma)k\cosh\frac{1}{2}(\pi-\gamma)k}\,\mathrm{d}k\right)$$

$$\xi_{hh}(\sigma)=\exp\left(\int_{-\infty}^\infty \frac{\sinh(\frac{1}{2}\pi)k(\mathrm{e}^{-\mathrm{i}k\sigma}-1)}{2k\sinh\frac{1}{2}\gamma k\sinh(\pi-\gamma)k\cosh\frac{1}{2}(\pi-\gamma)k}\,\mathrm{d}k\right).$$

Let us introduce new operators of creation–annihilation

$$Z_m(\sigma)=N_m(2\pi\tan\tfrac{1}{2}\gamma m p)^{-1/2}\psi_m(\sigma)\mathcal{R}_m(\sigma)$$

$$Z_m^+(\sigma)=N_m^{-1}(2\pi\tan\tfrac{1}{2}\gamma m p)^{1/2}\mathcal{R}_m^+(\sigma)\psi_m^{-1}(\sigma)$$

$$Z_h(\sigma)=N_h\psi_h(\sigma)\mathcal{H}(\sigma),\qquad Z_h^+(\sigma)=N_h^{-1}\mathcal{H}^+(\sigma)\psi_h^{-1}(\sigma).$$

The value of the constant N_h is not essential in the present paper.

Operators Z_m, Z_m^+, Z_h, Z_h^+ satisfy the same algebra as $\mathcal{R}_m$, $\mathcal{R}_m^+$, $\mathcal{H}$, $\mathcal{H}^+$ i.e. we have a new realisation of the Zamolodchikov–Faddeev algebra. The formula

$$Z_{m_1}^+(\sigma_1)\ldots Z_{m_k}^+(\sigma_k)\Omega_{\mathrm{ph}}=\prod_{j=1}^{k}(2\pi\tan\tfrac{1}{2}\gamma m p)^{1/2}N_{m_j}^{-1}$$
$$\times\prod_{i<j}\xi_{m_i m_j}(\sigma_i-\sigma_j)\mathcal{R}_{m_1}^+(\sigma_1)\ldots\mathcal{R}_{m_k}^+(\sigma_k)\Omega_{\mathrm{ph}}$$

and equation (1) allows us to express 'in' states in terms of operators Z_m^+.

The principal result of our previous paper (Smirnov 1984) is the derivation of the Gelfand–Levitan–Marchenko equation. We have introduced the operators Z, Z^+ to write this equation in an elegant form. We write down the equation in the continuous case. In the previous paper the lattice case was investigated but there are no problems with the continuous limit

$$g_-(x,\alpha)=\Omega_{\mathrm{ph}}^*+\frac{1}{2\pi\mathrm{i}}\sum_{m=1}^{(\pi/\gamma)-1}c_m\int_{-\infty}^\infty\exp\{\mathrm{i}[k_m(\sigma)-k_{m-1}(\sigma+\tfrac{1}{2}\mathrm{i}\gamma)]x\}$$
$$\times K_-[\alpha-\sigma+\tfrac{1}{2}\mathrm{i}\gamma(m-1)]g_-[x,\sigma+\mathrm{i}\pi-\tfrac{1}{2}\mathrm{i}\gamma(m-1)]$$
$$\times Z_{m-1}^+(\sigma+\tfrac{1}{2}\mathrm{i}\gamma)Z_m(\sigma)\,\mathrm{d}\sigma+(c_h/2\pi\mathrm{i})$$
$$\times\int_{-\infty}^\infty\exp\{\mathrm{i}[k_h(\sigma)-k_h(\sigma+\mathrm{i}\gamma)]x\}K_-(\alpha-\sigma+\tfrac{1}{2}\mathrm{i}\pi-\mathrm{i}\gamma)$$
$$\times g_-(x,\sigma+\tfrac{1}{2}\mathrm{i}\pi+\mathrm{i}\gamma)Z_h^+(\sigma+\mathrm{i}\gamma-\mathrm{i}\sigma)Z_h(\sigma)\,\mathrm{d}\sigma. \qquad (2)$$

Here $g_-(x,\alpha)$ is a covector function with the following asymptotics:

$$g_-(x,\alpha)\underset{\alpha\to-\infty}{\longrightarrow}\Omega_{\mathrm{ph}}^*\exp[-\mathrm{i}(2\gamma)^{1/2}u(x)+\tfrac{1}{2}pN_h\alpha].$$

N_h is the number of holes in the state with which we calculate the scalar product of $g_-(x, \alpha)$,

$$K_-(\sigma) = \tfrac{1}{2}p[\coth(\tfrac{1}{2}p\sigma) - 1],$$

$$c_m = \exp\left(-\int_{-\infty}^{\infty} \frac{\sinh(m-\frac{1}{2})\gamma k \sinh\frac{1}{2}\pi k}{k\cosh\frac{1}{2}(\pi-\gamma)k \sinh(\pi-\gamma)k}\, dk\right).$$

The value of c_h is not essential in the present paper. Solving equation (2) we determine the action of $\exp[-i(2\gamma)^{1/2}u(x)]$ on $\Omega^*_{\rm ph}$.

An equation similar to (2) can be obtained for the covector function $g_+(x, \alpha)$ with the asymptotics:

$$g_+(x, \alpha) \underset{\alpha\to\infty}{\longrightarrow} \Omega^*_{\rm ph} \exp[i(2\gamma)^{1/2}u(x) - \tfrac{1}{2}pN_h\alpha].$$

We have only to substitute K_- for

$$K_+(\sigma) = \tfrac{1}{2}p[\coth(\tfrac{1}{2}p\sigma) + 1].$$

In order to take into account the time dependence in the equations for $g_\pm(x, \alpha)$, we are to replace everywhere $k_m(\sigma)x$, $k_h(\sigma)x$ by $-p^\mu_m(\sigma)x_\mu$, $-p^\mu_h(\sigma)x_\mu$.

Consider the iterations of equations for $g_\pm(x, \alpha)$ in soliton-free subspace. The first step is to solve our equations in the subspace which contains only elementary bosons (without bound states):

$$g_\pm(x, t, \alpha) = \Omega^*_{\rm ph} \sum_{m=0}^{\infty} (c_1/2\pi i)^m \int K_\pm(\alpha - \sigma_m)$$
$$\times \prod_{j=2}^{m} K(\sigma_j - \sigma_{j-1} + i\pi) Z_1(\sigma_1) \ldots Z_1(\sigma_m)$$
$$\times \exp[-i \sum_{j=1}^{m} p^\mu_1(\sigma_j)x_\mu]\, d\sigma_1 \ldots d\sigma_m + \ldots$$

Dots denote terms with the annihilation operators of bound states and holes (solitons). We have for $\Omega^*_{\rm ph} \exp[\pm i(2\gamma)^{1/2}u(x, t)]$:

$$\Omega^*_{\rm ph} \exp[\pm i(2\gamma)^{1/2}u(x, t)]$$
$$= \Omega^*_{\rm ph} \sum_{m=0}^{\infty} (c_1/2\pi i)^m \int_{-\infty}^{\infty} \prod_{j=2}^{m} K(\sigma_j - \sigma_{j-1} + i\pi) Z_1(\sigma_1) \ldots Z_1(\sigma_m)$$
$$\times \exp\left(-i \sum_{j=1}^{m} p^\mu_1(\sigma_j)x_\mu\right) d\sigma_1 \ldots d\sigma_m + \ldots$$
$$= \Omega^*_{\rm ph} \sum_{m=0}^{\infty} (c_1/2\pi i)^m \int_{\sigma_m > \ldots > \sigma_1} X^\pm_{\underbrace{1,1,\ldots,1}_{m}}(\sigma_1, \ldots, \sigma_m)$$
$$\times Z_1(\sigma_1) \ldots Z_1(\sigma_m) \exp\left(-i \sum_{j=1}^{m} p^\mu_1(\sigma_j)x_\mu\right) d\sigma_1 \ldots d\sigma_m + \ldots$$

where

$$X^\pm_{1,1,\ldots,1}(\sigma_1, \ldots, \sigma_m) = \sum_{\rm perm} \prod_{j=1}^{m-1} K_\pm(\sigma_{\pi(j+1)} - \sigma_{\pi(j)} + i\pi)$$
$$\times \prod_{\substack{\pi(i) > \pi(j)\\ i<j}} S_{11}(\sigma_{\pi(i)} - \sigma_{\pi(j)}).$$

In the subspace containing the bound states one has to iterate our equations using the procedure proposed by Göckeler (1981). The final result is

$$\Omega^*_{\rm ph} \exp[\pm i(2\gamma)^{1/2}u(x,t)] = \Omega^*_{\rm ph} \sum_{m=0}^{\infty} \sum_{\substack{n_1+\ldots+n_k=m \\ n_j\leq(\pi/\gamma)-1}} (1/2\pi i)^k \times \prod_{l=1}^{k} \mu_{n_l} \int_{\sigma_k>\ldots\sigma_1} X^{\pm}_{n_1,\ldots,n_k}(\sigma_1,\ldots,\sigma_k) \exp\left(-i\sum_{j=1}^{k} p_{n_j}(\sigma_j)x_\mu\right) \times Z_{n_1}(\sigma_1)\ldots Z_{n_k}(\sigma_k)\,d\sigma_1\ldots d\sigma_k + \ldots, \tag{3}$$

where

$$\mu_m = c_1\ldots c_m \tan\tfrac{1}{2}\gamma p \ldots \tan\tfrac{1}{2}\gamma m p,$$

$$X^{\pm}_{n_1,\ldots,n_k}(\sigma_1,\ldots,\sigma_k) = X^{\pm}_{\underbrace{1,1,\ldots,1}_{n_1+n_2+\ldots+n_k}}[\sigma_1+\tfrac{1}{2}i\gamma(n_1-1), \times\sigma_1+\tfrac{1}{2}i\gamma(n_1-3),\ldots,\sigma_1-\tfrac{1}{2}i\gamma(n_1-1),\ldots,\sigma_k-\tfrac{1}{2}i\gamma(n_k-1)].$$

For general reasons it is clear that the expansion of $\Omega^*_{\rm ph}\cos(2\gamma)^{1/2}u[\Omega^*_{\rm ph}\sin(2\gamma)^{1/2}u]$ must contain only terms with an even (odd) number of elementary boson annihilation operators, i.e.

$$X^{+}_{n_1,\ldots,n_k}(\sigma_1,\ldots,\sigma_k) = (-1)^{n_1+\ldots+n_k} X^{-}_{n_1,\ldots,n_k}(\sigma_1,\ldots,\sigma_k). \tag{4}$$

In order to prove this equation we must solve a complicated combinatorial problem. We have proved (4) only for $n_1+n_2+\ldots n_k = 1, 2, 3, 4, 5$.

Let us write first as an example some $X^{\pm}$:

$$X^{\pm}_1(\sigma) = X^{\pm}(\sigma) = \pm 1, \qquad X^{\pm}_2(\sigma) = 1,$$

$$X^{\pm}_{1,1}(\sigma_1,\sigma_2) = \sinh p\sigma_{21}/\sinh\tfrac{1}{2}p(\sigma_{21}-i\gamma)\cosh\tfrac{1}{2}p(\sigma_{21}+i\gamma),$$

$$X^{\pm}_{1,1,1}(\sigma_1,\sigma_2,\sigma_3) = \pm\prod_{i>j}\frac{\sinh\frac{1}{2}p\sigma_{ij}}{\sinh\frac{1}{2}p(\sigma_{ij}-i\gamma)\cosh\frac{1}{2}p(\sigma_{ij}+i\gamma)} \times(4\cosh\tfrac{1}{2}p\sigma_{21}\cosh\tfrac{1}{2}p\sigma_{31}\cosh\tfrac{1}{2}p\sigma_{32}+2\cos^2\tfrac{1}{2}\gamma p), \tag{5}$$

$$X^{\pm}_{2,1}(\sigma_1,\sigma_2) = \frac{\sinh\frac{1}{2}p(\sigma_{21}+i\gamma)(\cosh p\sigma_{21}+2\cos\frac{1}{2}p\gamma)}{\cosh\frac{1}{2}p(\sigma_{21}+i\gamma)\sinh\frac{1}{2}p(\sigma_{21}-\frac{3}{2}i\gamma)\cosh\frac{1}{2}p(\sigma_{21}+\frac{3}{2}i\gamma)},$$

$$\sigma_{ij} \equiv \sigma_i - \sigma_j.$$

Using the expansion (3) one can obtain any form factor of $\exp[\pm i(2\gamma)^{1/2}u(x,t)]$ corresponding to any number of elementary bosons and their bound states:

$$\Omega^*_{\rm ph}\exp[\pm i(2\gamma)^{1/2}u(x,t)]|p_1,n_1;\ldots;p_k,n_k\rangle_{\rm in} = \exp\left(i\sum_{j=1}^{k} p^{\mu}_{n_j}(\sigma_j)x_\mu\right)F^{\pm}_{n_1,\ldots,n_k}(p_1,\ldots,p_k), \tag{6}$$

$$F_{n_1,\ldots,n_k}(p_1,\ldots,p_k) = \prod_{j=1}^{k}(2\pi\tan\tfrac{1}{2}\gamma n_j p)^{1/2}\mu_{n_j} \times\prod_{p>q}\xi_{n_{j_p}n_{j_q}}(\sigma_{j_p}-\sigma_{j_q})X^{\pm}_{n_1,\ldots,n_k}(\sigma_{j_1},\ldots,\sigma_{j_k})$$

$$\sigma_{j_k} > \sigma_{j_{k-1}} > \ldots > \sigma_{j_1}.$$

Now let us consider the analytical properties of $\xi_{mn}(\sigma)$. From the definition of ξ_{mn} it is clear that $\xi_{mn}(\sigma)=\xi_{mn}(-\sigma)=\xi_{nm}(\sigma)$. It can be shown that $\xi_{mn}(\sigma)$ has analytical continuation into the strip $0\leqslant \mathrm{Im}\,\sigma\leqslant \pi-\gamma$. It has one pole at the point $\sigma=\mathrm{i}\pi-\mathrm{i}\gamma$ if $m=n$ and has no poles if $m\neq n$. It also has $\min(m,n)$ zeros at points $\sigma=\mathrm{i}\pi-\frac{1}{2}\mathrm{i}\gamma(m+n-2j+4)$, $j=1,\ldots,\min(m,n)$. On the line $\mathrm{Im}\,\sigma=\pi-\gamma$ one has:

$$\xi_{mn}(\sigma+\mathrm{i}\pi-\mathrm{i}\gamma)=\prod_{j=1}^{\min(m,n)}\coth\tfrac{1}{2}p[\sigma+\mathrm{i}\pi+\tfrac{1}{2}\mathrm{i}\gamma(|m-n|+2j-2)]$$

$$\times\coth\tfrac{1}{2}p[\sigma-\tfrac{1}{2}\mathrm{i}\gamma(m+n-2j)]$$

$$\times\exp\left(-\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}k\sigma}\sinh(\frac{1}{2}\pi-\gamma)k\sinh\frac{1}{2}\gamma mk\sinh\frac{1}{2}\gamma nk}{k\sinh(\pi-\gamma)k\sinh\frac{1}{2}\gamma k\cosh\frac{1}{2}(\pi-\gamma)k}\,\mathrm{d}k\right).$$

Now we want to show that the two-particle form factor satisfies all requirements proposed by Karowsky and Weisz (1978). It follows from (5) and (6) that

$$F_{1,1}(p_1,p_2)=\frac{\sinh pz}{\sinh\frac{1}{2}p(z-\mathrm{i}\gamma)\cosh\frac{1}{2}p(z+\mathrm{i}\gamma)}\xi_{1,1}(z)\equiv f(z)$$

$$z\equiv|\sigma_{21}|.$$

The function $f(z)$ has analytical continuation into the strip $0\leqslant\mathrm{Im}\,z\leqslant\pi-\gamma$. The possibility of the production of a two-particle bound state is manifested in the existence of the simple pole at the point $z=\mathrm{i}\gamma$ which is the only singularity of $f(z)$ in the strip. Besides

$$f(z)=f(-z)S_{11}(z)$$

$$f(\mathrm{i}\pi-\mathrm{i}\gamma-z)=f(\mathrm{i}\pi-\mathrm{i}\gamma+z),$$

in complete agreement with Karowsky and Weisz (1978).

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